Arbitrage

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BUSI 448: Investments

RICE | BUSINESS

Where are we?

Last time:

- Treasury market basics
- Term structure
- Spot rates

Today:

- Arbitrage

Arbitrage and pricing

No-arbitrage Pricing

- Arbitrage is a trade that generates positive profits in some state of the world and generates losses in **no** state of the world.
- Bootstrapping is based on the principle of **no-arbitrage pricing**.
  - First, assume a pricing relationship between two or more assets
  - Second, try to construct an arbitrage from that relation
  - If you can, then the assumed relationship cannot be true (or you should trade!)

Law of One Price

- Two assets that generate the same cash flows should have the same price.
- We can try to form a **replicating portfolio** with the same cash flows as the security we are trying to value.

Example #1

Example

Suppose we have a 1-year bond with cash flows of $15 in 6 months and $115 in one year that costs $125.

- What is the spot-rate implied price of this bond?
- Can we replicate the cash flows of this bond with a portfolio of A, B, C, D?

From last time: Bonds A, B, C, D

Bond	Price	Coupon Rate	Maturity	Face Value
A	97.5	0%	0.5	100
B	95	0%	1.0	100
C	955	2.5%	1.5	1,000
D	1,000	5.75%	2	1,000

Replicating portfolio

- We only need to use bonds A and B since the target bond is a one-year bond.
- Let x_A denote the amount invested in bond A and x_B be the amount invested in bond B.
- We want to solve the following system of equations (one for each time period with payments for the 1-year target bond):

$$x_A \cdot A_1 + x_B \cdot B_1 = 15$$

$$x_A \cdot A_2 + x_B \cdot B_2 = 115$$

Replicating portfolio

$$x_A \cdot 100 + x_B \cdot 0 = 15$$

$$x_A \cdot 0 + x_B \cdot 100 = 115$$

Solving for x_A and x_B:

$$x_A = \frac{15}{100} = 0.15$$

$$x_B = \frac{115}{100} = 1.15$$

Market price versus replicating portfolio value

- The replicating portfolio has a value of $123.875
- The actual 1-year bond price is $125.
- Can we construct an arbitrage trade?

The arbitrage trade

- Sell the (15,115) 1-year bond
- Buy the replicating portfolio
- For each unit of the 1-year bond:
  - buy 0.15 of A
  - buy 1.15 of B
- Profit of $1.125 per $100 face in 1-year bond

Example #2

Replication with coupon bonds

What did we do above?

- We solved a system of equations to create a portfolio of bonds that each mature at different periods in order to replicate the cash flows of our 'target' bond.
- Can we form replicating portfolios using coupon bonds as well?

YES!

Example (2-yr annual bond)

- Suppose the price of the 2-year bond from last time was actually $1,050. Let’s call this bond V.
  - 2-year annual bond with face of 1,000 and coupon rate of 10%.
- Can we construct an arbitrage trade?
- Again, the key is to construct a portfolio of bonds A, B, C, and D that **replicates** the CFs of the 2-year annual bond.

Replicating the two-year bond CFs

$$x_A \cdot A_1 + x_B \cdot B_1 + x_C \cdot C_1 + x_D \cdot D_1 = V_1$$
$$x_A \cdot A_2 + x_B \cdot B_2 + x_C \cdot C_2 + x_D \cdot D_2 = V_2$$
$$x_A \cdot A_3 + x_B \cdot B_3 + x_C \cdot C_3 + x_D \cdot D_3 = V_3$$
$$x_A \cdot A_4 + x_B \cdot B_4 + x_C \cdot C_4 + x_D \cdot D_4 = V_4$$

Plugging in the cash flows for each bond

$$
\begin{aligned}
x_A \cdot 100 &+ x_B \cdot 0 &+ x_C \cdot 12.5 &+ x_D \cdot 28.75 &= 0 \\
x_A \cdot 0 &+ x_B \cdot 100 &+ x_C \cdot 12.5 &+ x_D \cdot 28.75 &= 100 \\
x_A \cdot 0 &+ x_B \cdot 0 &+ x_C \cdot 1012.5 &+ x_D \cdot 28.75 &= 0 \\
x_A \cdot 0 &+ x_B \cdot 0 &+ x_C \cdot 0 &+ x_D \cdot 1028.75 &= 1100
\end{aligned}
$$

Solving for the position sizes x

- The intuition is the same as the two-period bond, but the math is more annoying.
- Option #1: grind through the algebra
  - what if there are more than 4 periods?!?
- Option #2: let python solve it for us with matrices

Replicating portfolio w/ matrices

The system of equations above can be written in matrix notation as:

$$\mathbf{x} \cdot \mathbf{CF} = \mathbf{cf}$$

- $\mathbf{x} = \begin{bmatrix} x_A & x_B & x_C & x_D \end{bmatrix}$
- $\mathbf{CF} = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \\ C_1 & C_2 & C_3 & C_4 \\ D_1 & D_2 & D_3 & D_4 \end{bmatrix}$
- $\mathbf{cf} = \begin{bmatrix} V_1 & V_2 & V_3 & V_4 \end{bmatrix}$

Solving for the position sizes x in python

We can solve for $\mathbf{x}$ in the system of equations

$$\mathbf{x} \cdot \mathbf{CF} = \mathbf{cf}$$

in python using `cf @ np.linalg.inv(CF)`!

Market price versus replicating portfolio value

- The replicating portfolio has a value of $1,076.82
  - look familiar?
- The actual 2-year bond price is $1,050.
- Can we construct an arbitrage trade?

The arbitrage trade

- Buy the 2-year bond
- Sell the replicating portfolio
- For every unit of the 2-year bond:
  - buy 0.304 of A
  - sell 0.696 of B
  - buy 0.030 of C
  - sell 1.069 of D
- Profit of $26.82 for each unit of 2-year bond bought

For next time: Markets, Trading, & Adverse Selection