# Optimal Portfolios: Borrowing Frictions

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BUSI 448: Investments



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# Where are we?

**Last time**:

- Capital Allocation: Risk-free + Risky
- Preferences over risk and return
- Optimal portfolios

**Today**:

- Borrowing frictions

# Borrowing frictions

# Leverage constraints

Many investors (like me!) cannot borrow at the same rate at which they can lend.

In this case, the capital allocation line is not a straight line.

For most investors:

$$r^{\text{borrow}} > r^{\text{saving}}$$

# Kinked Capital Allocation Lines

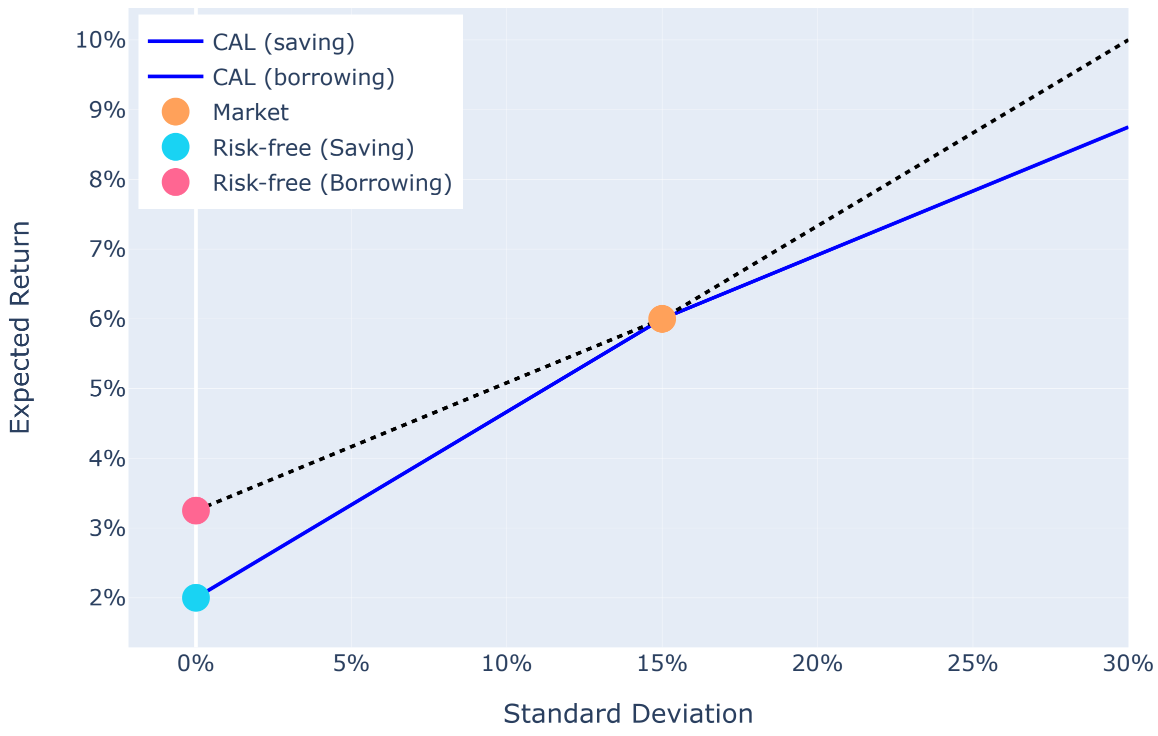
- For portfolios with some risk-free saving:

$$E[r_p] = r_f^{\text{saving}} + \left[ \frac{E[r_{\text{risky}}] - r_f^{\text{saving}}}{\text{sd}[r_{\text{risky}}]} \right] \cdot \text{sd}[r_p]\,.$$

- For portfolios with borrowing, the capital allocation line has a lower slope:

$$E[r_p] = r_f^{\text{borrow}} + \left[ \frac{E[r_{\text{risky}}] - r_f^{\text{borrow}}}{\text{sd}[r_{\text{risky}}]} \right] \cdot \text{sd}[r_p]\,.$$

# Kinked Capital Allocation Lines



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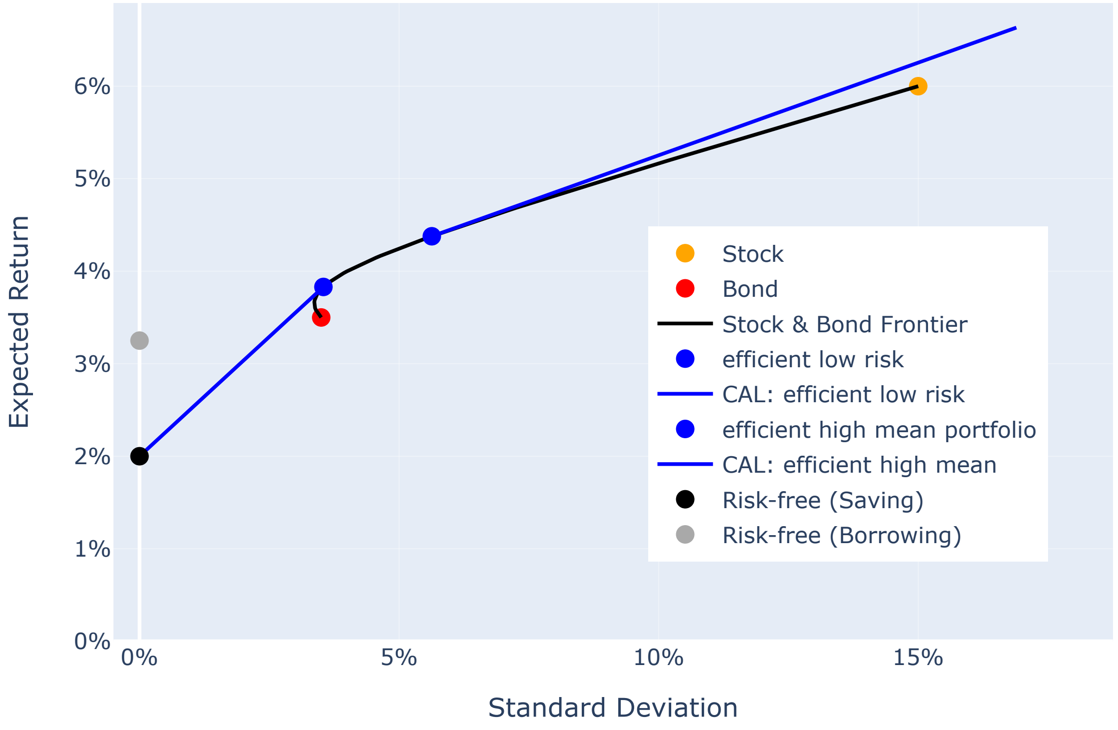
# Optimal portfolios with leverage frictions



# Capital allocation with leverage frictions

- Where do investors with different risk aversions choose to invest when faced with this investment opportunity set?
- The answer depends on the investor's risk aversion and the reward-risk ratios of the efficient low risk and high mean portfolios.

# Capital allocation with leverage frictions

**High risk aversion investors** invest in the efficient low risk portfolio and save:

$$w^*_{\text{low}} = \frac{E[r_{\text{low}} - r_f^{\text{saving}}]}{A \cdot \text{var}_{\text{low}}}.$$

**Low risk aversion investors** invest in the efficient high mean portfolio and borrow:

$$w^*_{\text{high}} = \frac{E[r_{\text{high}} - r_f^{\text{borrow}}]}{A \cdot \text{var}_{\text{high}}}.$$

# Capital allocation with leverage frictions

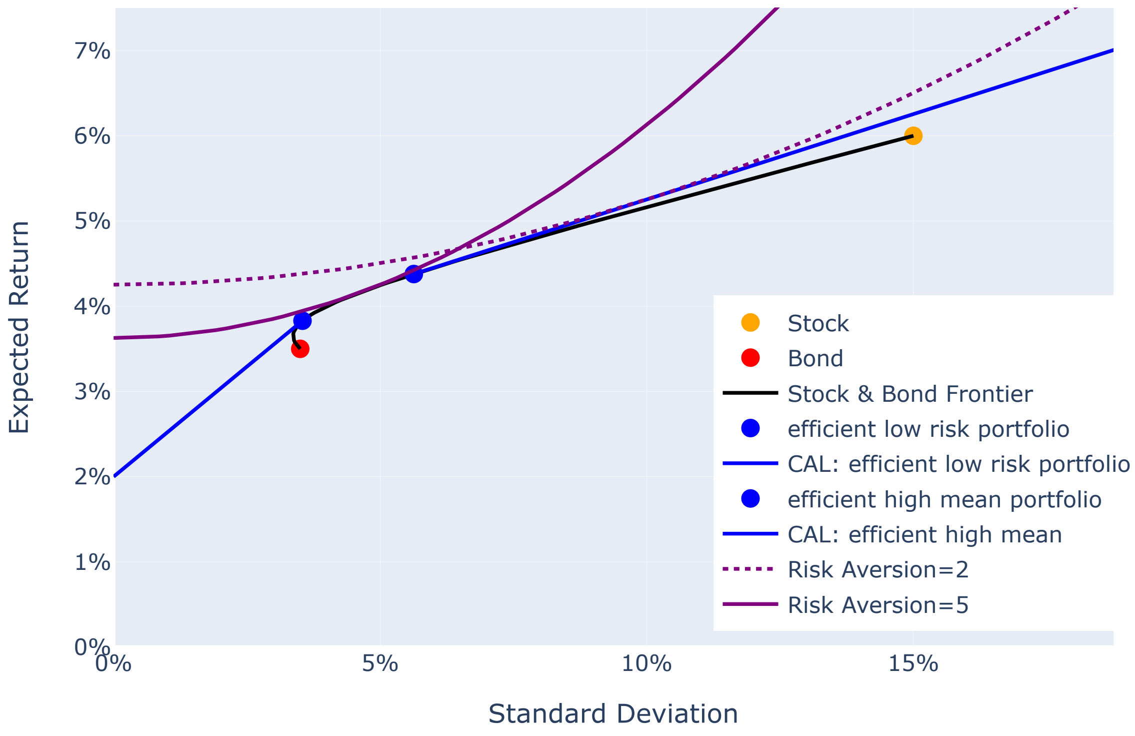
**Intermediate risk aversion investors** invest in risky assets only.

- Can express as a two-asset portfolio of the efficient low and high risk portfolios.
- The optimal weight $a^*$ in the low-risk portfolio is:

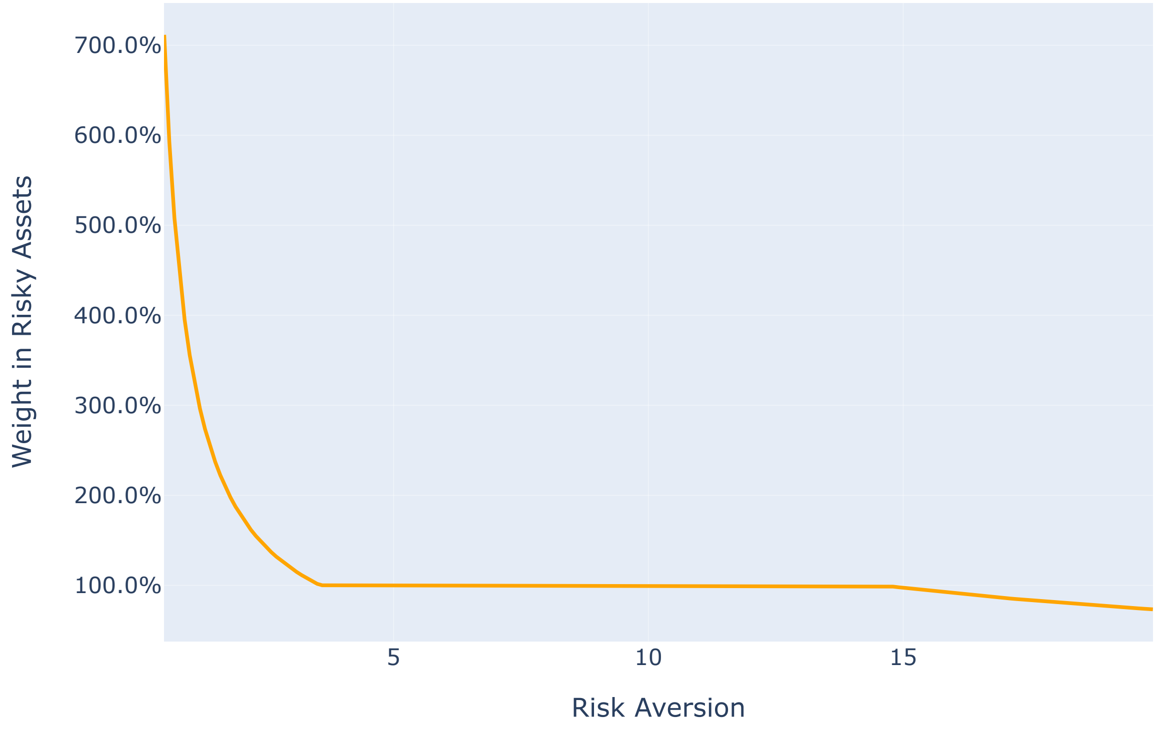
$$a^* = \frac{E[r_{\text{low}} - r_{\text{high}}] - A(\text{cov}[r_{\text{low}}, r_{\text{high}}] - \text{var}[r_{\text{high}}])}{A(\text{var}[r_{\text{low}}] + \text{var}[r_{\text{high}}] - 2\text{cov}[r_{\text{low}}, r_{\text{high}}])},$$

- Note: $\text{cov}[r_{\text{low}}, r_{\text{high}}] = w'_{\text{low}} V w_{\text{high}}$, where $w_{\text{low}}$ and $w_{\text{high}}$ are the weights in the underlying risky assets for the efficient low-risk and high-mean portfolios, respectively.

# Capital allocation with leverage frictions



# Risky asset allocation with leverage frictions



# Capital allocation: two solution methods

- Method #1: Find risk aversion thresholds that represent low- and high-risk portfolios
- Method #2: Directly maximize mean-variance utility using all assets, including risk-free savings and borrowings

# Method #1: Risk aversion thresholds

We can find the risk aversion thresholds for savings and borrowing by setting risky asset allocation $w^* \leq 1$ (savings) or $w^* \geq 1$ (borrowing) in the capital allocation expressions and solving for risk aversion.

- **Upper risk aversion threshold**: some savings if

$$A \geq \frac{E[r_{\text{low}} - r_f]}{\text{var}(r_{\text{low}})}.$$

- **Lower risk aversion threshold**: some borrowing if

$$A \leq \frac{E[r_{\text{high}} - r_f]}{\text{var}(r_{\text{high}})}.$$

# Method #2: Direct optimization of utility

The optimal portfolio for investor with risk aversion $A$ solves:

$$\max_{w_{\text{saving}}, w_{\text{borrow}}, w_1, w_2, \ldots, w_N} E[r_p] - 0.5 \cdot A \cdot \text{var}[r_p]$$

subject to the constraints

$$w_{\text{saving}} + w_{\text{borrow}} + \sum_i w_i = 1,$$

$$w_{\text{saving}} \geq 0,$$

$$w_{\text{borrow}} \leq 0.$$

- We need to augment the expected return vector and covariance matrix with elements for the savings and borrowing assets.

# Mapping to `cvxopt.solvers.qp`

Recall the `cvxopt.solvers.qp` function's general form:

$$\min_{w} \frac{1}{2} w' Q w + p' w$$
$$\text{subject to } Gw \leq h$$
$$Aw = b$$

- $0.5 w' Q w$ captures $0.5 \cdot A \cdot \text{var}[r_p]$
- $p' w$ captures $-E[r_p]$
- $Gw \leq h$ captures only positive saving and negative borrowing
- $Aw = b$ is the fully invested constraint

# Python implementation

```python
def opt_allocation2(means, cov, rf_save, rf_borrow, risk_aversion):
    n=len(means)
    Q = np.zeros((n + 2, n + 2))
    Q[2:, 2:] = risk_aversion * cov
    Q = matrix(Q, tc="d")
    p = np.array([-rf_save, -rf_borrow] + list(-means))
    p = matrix(p, (n + 2, 1), tc="d")
    # Constraint: saving weight positive, borrowing weight negative
    G = np.zeros((2, n + 2))
    G[0, 0] = -1
    G[1, 1] = 1
    G = matrix(G, (2, n+2), tc="d")
    h = matrix([0, 0], (2, 1), tc="d")
    # Constraint: fully-invested portfolio
    A = matrix(np.ones(n+2), (1, n+2), tc="d")
    b = matrix([1], (1, 1), tc="d")
    sol = Solver(Q, p, G, h, A, b)
    if sol["status"] == "optimal":
        wgts_optimal = np.array(sol["x"]).flatten()
```



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# Learn Investments Dashboard resources

## Optimal allocation with different rates

# For next time: Short-sale constraints